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**PRICING EUROPEAN OPTIONS BASED ON  
FOURIER-COSINE SERIES EXPANSIONS**

Mestrado in Mathematical Finance

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# Resumo

Nesta tese, apresenta-se e discute-se um método de avaliação de opções, para opções europeias, baseado na série de Fourier de cossenos e chamado “COS method”. A ideia fundamental do método reside na relação estreita entre a função característica e a série dos coeficientes da expansão de Fourier dos cossenos da função densidade. Este método de avaliação, proposto por Fang e Oosterlee [10], é aplicável a todos os processos dos ativos subjacentes para os quais a função característica é conhecida. Deste modo, o método é aplicável a vários tipos de contratos sobre opções, tais como os da classe dos modelos exponenciais de Lévy e o modelo de Heston (1993). Iremos provar que, na maioria dos casos, a convergência do método COS é exponencial.

**Palavras-Chave:** Avaliação de opções, opções europeias, expansão de Fourier de cossenos.

# Abstrat

In this thesis, we present and discuss an option pricing method for European options based on the Fourier-Cosine series called COS method. The key insight is in the close relation between the characteristic function and the series coefficients of the Fourier-Cosine expansion of the density function. This pricing method, proposed by Fang and Oosterlee (2008), is applicable to all underlying asset processes for which the characteristic function is known. The method is thus applicable to many types of option contracts, such as the class of exponential Lévy models and the Heston (1993) model. We will show that, in most cases, the convergence rate of the COS method is exponential.

**Keywords:** Option Pricing, European Options, Fourier Cosine Expansion.

# Contents

List of Figures	vi
List of Tables	vii
<b>1 Introduction</b>	<b>1</b>
1.1 Literature Review . . . . .	1
1.2 Framework . . . . .	2
1.3 Overview of This Thesis . . . . .	4
<b>2 Models</b>	<b>6</b>
2.1 Definitions . . . . .	6
2.2 Geometric Brownian Motion Process . . . . .	10
2.3 The Variance Gamma Process . . . . .	11
2.3.1 VG as Brownian Motion with a Drift . . . . .	12
2.3.2 VG as a Difference of Gamma Processes . . . . .	14
2.4 Heston Model . . . . .	15
<b>3 The COS Method</b>	<b>20</b>
3.1 Fourier Integrals and Cosine Series . . . . .	20
3.1.1 Inverse Fourier Integral via Cosine Expansion . . . . .	20
3.2 Pricing European Options . . . . .	22
3.2.1 Coefficients $V_k$ for Plain Vanilla Options . . . . .	24
3.2.2 Formula for Exponential Lévy Processes and the Heston Model . . . . .	25
<b>4 Numerical Results</b>	<b>27</b>
4.1 Experimental Setup . . . . .	27
4.2 GBM . . . . .	28
4.3 VG . . . . .	29
4.4 Heston . . . . .	30
<b>5 Conclusions</b>	<b>33</b>



# List of Figures

4.1	Recovered density function of the GBM model; $K = 100$ , with other parameters as in (4.3)	28
4.2	COS error convergence for pricing European call options under the GBM model.	29
4.3	Recovered density functions for the VG model and two maturity dates, with parameters as in (4.4).	30
4.4	COS error convergence for pricing European call options under the VG model.	31
4.5	Recovered density function of the Heston model, with parameters as in (4.5).	32



# List of Tables

4.1	Error convergence and CPU time using the COS method for European calls under GBM, with parameters as in (4.3); $K=80, 100, 120$ ; reference values= $20.799226309 \dots, 3.659968453 \dots$ and $0.04477814$ , respectively.	29
4.2	Convergence of the COS method for a call under the VG model with parameters as in (4.4).	30
4.3	Error convergence and CPU times for the COS method for calls under the Heston model with $T = 1$ , with parameters as in (4.5); reference value= $5.785155450$ .	31
4.4	Error convergence and CPU times for the COS method for calls under the Heston model with $T = 10$ , with parameters as in (4.5); reference value= $22.318945791$ .	31
A.1	Cumulants for the GBM, VG and Heston models; and $w$ , the drift correction term, which satisfies $\exp(-wt) = \phi(-i, t)$ .	35

# Chapter 1

## Introduction

In this thesis, we will present an option pricing method for European options based on Fourier-cosine expansions in the context of numerical integration. This initial chapter starts, in Section 1.1, with a literature review, presenting afterward the goal of this thesis. Next, in Section 1.2, the framework of this work is presented, namely the options definitions are given and the pricing problem under analysis is described. Finally, an overview of this thesis is provided in Section 1.3.

### 1.1 Literature Review

When valuing and risk-managing exotic derivatives, the speed and efficiency of the methods being used are extremely important. In fact, the success or failure of financial institutions is somehow dependent on the swift and accurate application of these methods. The numerical methods commonly used for such purposes can be briefly classified into three groups: partial-integro differential equation (PIDE) methods, Monte Carlo simulation, and numerical integration methods. Efficient numerical methods are required to rapidly price complex contracts and calibrate financial models. During calibration, i.e., when fitting model parameters of the stochastic asset processes to market data, we typically need to price European options at a single spot price, with many different strike prices, very quickly.

The probability density function appearing in the integration in the original pricing domain is not known for many relevant asset processes. However, its Fourier transform, the characteristic function, is often available. The integration methods are used for calibration purposes whenever the characteristic function of the asset price process is known analytically. A wide class of examples arises when the dynamics of the log price is given by an infinitely divisible process of independent increments. The characteristic function then arises naturally from the Lévy-Khintchine representation for such processes. Among this class of processes, we have, for instance, the VG process [18]

and the Carr-Madan method [7], which is one of the best known in this class. Characteristic functions have also been used in the pure diffusion context with stochastic volatility by Heston [11] and with stochastic interest rates by Bakshi and Chen [2]. Finally, they have been used in the CONV method [15]. In the Fourier domain, it is then possible to price various derivative contracts efficiently.

As previously mentioned, an important aspect of research in computational finance is to further increase the performance of the pricing methods. Quadrature rule based techniques are not of the highest efficiency when solving Fourier transformed integrals. As the integrands are highly oscillatory, a relatively fine grid has to be used for satisfactory accuracy with the Fast Fourier Transform.

In this work, we will focus on a numerical method, called the COS method, proposed by Fang and Oosterlee [10], which shows that this method can further improve the speed of pricing plain vanilla and some exotic options. Furthermore, the COS method offers a highly efficient way to recover the density from the characteristic function, which is of importance to several financial applications, such as calibration, the computation of forward starting options, or static hedging.

## 1.2 Framework

### Options Definitions and Terminology

Before we introduce the problem addressed in this work, we begin by presenting some definitions and terminology associated with options, as defined in [12].

There are two types of options. A *call option* gives the holder the right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise price* or *strike price*; the date in the contract is known as the *expiration date* or *maturity*. According to the date(s) in which we can exercise the option, there are American and European options. American options can be exercised at any time up to the expiration date, while European options can be exercised only on the expiration date itself.<sup>1</sup> An option is a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables, such as stocks, indices, currencies, bonds, or other derivatives. In this thesis we will be focused on stock options. It should be emphasized that an option gives the holder the right to do something. The holder does not have to exercise this right. This is what distinguishes options from forwards and futures, where the holder is obligated to buy or sell the underlying asset. Whereas it costs nothing to enter into a forward or futures contract, there is a cost to acquiring an option, usually called *option premium*.

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<sup>1</sup>Note that the terms *American* and *European* do not refer to the location of the option or the exchange.

During the lifetime of an option, the underlying asset price varies and therefore the option is also classified regarding the relation between the current asset price,  $S$ , and the contract strike price,  $K$ . For instance, a call where  $S > K$  is referred to as being in-the-money (ITM), if the prices are the same,  $S = K$ , it is at-the-money (ATM) and out-of-the-money (OTM) when  $S < K$ .

Throughout this thesis, we will consider  $(S_t)_{t \in [0, T]}$  the price of an asset modelled as a stochastic process on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  is the set of all outcomes that are possible.  $\mathcal{F}$  is a *sigma-algebra* containing all sets for which we want to assess, where the filtration  $\mathcal{F} = \{\mathcal{F}_t, t \in [0, T]\}$  satisfies the usual conditions.  $\mathbb{P}$  is the physical measure which gives the probability that an event contained in the set of  $\mathcal{F}$  might occur. Note that, under the hypothesis of absence of arbitrage, there exists a measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  under which the discounted prices of all traded financial assets are  $\mathbb{Q}$ -martingales.

### Pricing Problem

The pricing problem under analysis in this thesis is the valuation of European options. To this end, we will denote the option value at time  $t$ , by  $v_t$ , which attending to the above definition of a call and put option reads

$$v_T = \begin{cases} (S_T - K)^+ & \text{for a call,} \\ (K - S_T)^+ & \text{for a put.} \end{cases} \quad (1.1)$$

Now, focusing on a European call option, the *time*  $- t$  price of a European call option on a non-dividend paying stock with spot price  $S_t$ , when the strike is  $K$  and the time to maturity is  $\tau = T - t$ , it is the discounted expected value of the payoff under the risk-neutral measure  $\mathbb{Q}$ ,

$$v^{call}(S, K, T) = e^{-r\tau} \mathbf{E}^{\mathbb{Q}}[(S_T - K)^+] \quad (1.2)$$

$$\begin{aligned} &= e^{-r\tau} \mathbf{E}^{\mathbb{Q}}[(S_T - K) \mathbf{1}_{\{S_T > K\}}] \\ &= e^{-r\tau} \mathbf{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T > K\}}] - K e^{-r\tau} \mathbf{E}^{\mathbb{Q}}[\mathbf{1}_{\{S_T > K\}}] \end{aligned} \quad (1.3)$$

where  $\mathbf{1}$  is the indicator function. In Equation (1.3), the expected value  $E^{\mathbb{Q}}[\mathbf{1}_{\{S_T > K\}}]$  is the probability of the call expiring in-the-money under the measure  $\mathbb{Q}$ . We can therefore write

$$\mathbf{E}^{\mathbb{Q}}[\mathbf{1}_{\{S_T > K\}}] = \mathbb{Q}(S_T > K).$$

Evaluating  $e^{-r\tau} \mathbf{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T > K\}}]$  in (1.3) requires changing the original measure  $\mathbb{Q}$  to

another measure  $\mathbb{Q}^S$ . Consider the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{B_T/B_t}{S_T/S_t} = \frac{\mathbf{E}^{\mathbb{Q}}[e^{x_T}]}{e^{x_T}} \quad (1.4)$$

where

$$B_t = \exp\left(\int_0^t r \, du\right) = e^{rt}.$$

In (1.4), we have written  $S_t e^{r(T-t)} = E^{\mathbb{Q}}[e^{x_T}]$ , since under  $\mathbb{Q}$  assets grow at the risk-free rate,  $r$ . The first expectation in (1.3) can therefore be written as

$$\begin{aligned} e^{-r\tau} \mathbf{E}^{\mathbb{Q}}[S_T \mathbf{1}_{\{S_T > K\}}] &= S_t \mathbf{E}^{\mathbb{Q}}\left[\frac{S_T/S_t}{B_T/B_t} \mathbf{1}_{\{S_T > K\}}\right] \\ &= S_t E^{\mathbb{Q}^S}\left[\frac{S_T/S_t}{B_T/B_t} \mathbf{1}_{\{S_T > K\}} \frac{d\mathbb{Q}}{d\mathbb{Q}^S}\right] \\ &= S_t E^{\mathbb{Q}^S}[\mathbf{1}_{\{S_T > K\}}] \\ &= S_t \mathbb{Q}^S(S_T > K). \end{aligned} \quad (1.5)$$

This implies that the European call price of Equation (1.3) can be written in terms of both measures as

$$\begin{aligned} v^{call}(S, K, T) &= S_t \mathbb{Q}^S(S_T > K) - K e^{-r\tau} \mathbb{Q}(S_T > K) \\ &= S_t P_1 - K e^{-r\tau} P_2, \end{aligned} \quad (1.6)$$

where  $P_1 := \mathbb{Q}^S(S_T > K)$  and  $P_2 := \mathbb{Q}(S_T > K)$ . The measure  $\mathbb{Q}$  uses the bond  $B_t$  as the numeraire, while the measure  $\mathbb{Q}^S$  uses the stock price  $S_t$ .

Using the well-known *put-call parity*, the price of a European put option on the same stock with the same strike and maturity, reads

$$v^{put}(S, K, T) = v^{call}(S, K, T) + K e^{-r\tau} - S_t \quad (1.7)$$

which is valid for any model (for a proof, please see [12]).

### 1.3 Overview of This Thesis

This thesis is organized as follows. In Chapter 2, we begin by introducing the main definitions and properties of some basic stochastic processes, which serve as the building blocks for more complicated processes. Then, we present the models considered in this thesis, namely: the geometric Brownian motion, a diffusion model; the Variance Gamma model, a pure jump model, and the Heston (1993) model, a stochastic volatility model. Next, in Chapter 3, we describe the option pricing method for European options based on the Fourier-Cosine series, the COS method. The key insight is in

the close relation between the characteristic function and the series coefficients of the Fourier-Cosine expansion of the density function. Special attention is given to the implementation details.

In Chapter 4, the performance of the method is evaluated in terms of speed and accuracy by pricing European options. Based on the inversion technique presented in Chapter 3, the underlying density function for each individual experiment is also recovered. Finally, Chapter 5 concludes this thesis by outlining the main contributions of this method to the options pricing field and puts forth a few possibilities for future research.

# Chapter 2

## Models

This section presents some models which will be used as the driving stochastic processes of the asset returns. We begin Section 2.1 by presenting some definitions and properties of such processes. Section 2.2 presents the geometric Brownian motion, which is a diffusion model. Next, in Section 2.3, another Lévy process is presented, a three parameter generalization of the Brownian motion, the so-called variance gamma model. Finally, Section 2.4 introduces a stochastic volatility model, the Heston (1993) model.

### 2.1 Definitions

In the modelling of financial markets, especially in the stock market, the Brownian Motion (BM) plays a significant role in building a statistical model. Let us start by defining a standard Brownian motion.

**Definition 2.1.1 (Brownian Motion).** *A standard **Brownian Motion** (or Wiener process)  $W = \{W_t, t \geq 0\}$ , is a real valued stochastic process defined on a filtered probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$  satisfying:*

1.  $W_0 = 0$ , almost surely; that is,  $\mathbb{P}(W_0 = 0) = 1$ .
2.  $W$  has independent increments: for every increasing sequence of times  $t_0 \dots t_n$ , the random variables  $W_{t_0}, W_{t_1} - W_{t_0}, \dots, W_{t_n} - W_{t_{n-1}}$  are independent.
3.  $W$  has stationary increments, i.e., the law of  $W_{t+h} - W_t$  does not depend on  $t$ .
4.  $W_t \sim \text{Normal}(0, t)$ . Its increments follow a Gaussian distribution with mean 0 and variance  $t$ .

**Theorem 2.1.1 (Standard Brownian motion).** *A standard Brownian motion process  $(W_t)$  defined on a filtered probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$  satisfies the following conditions:*

1. The process is stochastically continuous:  $\forall \epsilon > 0, \lim_{h \rightarrow 0} (\mathbb{P}(|W_{t+h} - W_t| \geq \epsilon)) = 0$ .
2. Its sample path (trajectory) is continuous in  $t$ .

*Proof:* Please see [13].

This last property, plays a crucial role in the properties of diffusion models. Cont and Tankov [8] show that this property is actually not robust in the presence of jumps in asset price dynamics. Thus, jump processes also have an important place in modeling financial markets. The fundamental pure jump process is the Poisson process. Its definition is given as follows, based on [8]:

**Definition 2.1.2 (Poisson process).** Let  $(\tau_i)_{i \geq 1}$  be a sequence of independent exponential random variables with parameter  $\lambda$  and  $T_n = \sum_{i=1}^n \tau_i$ . The process  $(N_t, t \geq 0)$  defined by

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}} \quad (2.1)$$

is called a Poisson process with intensity  $\lambda$ .

Below we present some properties of the Poisson process:

**Proposition 2.1.1.** Let  $(N_t)_{t \geq 0}$  be a Poisson process with intensity  $\lambda > 0$ , and let  $0 = t_0 < t_1 < \dots < t_n$  be given. Then

1.  $N_t$  has stationary increments: the law of  $N_{t+h} - N_t$  does not depend on  $t$ ;
2.  $N_t$  has independent increments: for every increasing sequence of times  $t_0 \dots t_n$ , the random variables  $N_{t_0}$ ,  $N_{t_1} - N_{t_0}$ , ...,  $N_{t_n} - N_{t_{n-1}}$  are independent.
3.  $N_t$  is stochastically continuous:  $\forall \epsilon > 0, \lim_{h \rightarrow 0} (\mathbb{P}(|N_{t+h} - N_t| \geq \epsilon)) = 0$ .
4. The increments of  $N$  are homogeneous: for any  $t > s$ ,  $N_t - N_s$  has the same distribution as  $N_{t-s}$ .

*Proof:* Please see Cont and Tankov [8].

All jumps of a Poisson process are of size one. The jumps of  $N_t$  occur at times  $T_i$ . A *compound Poisson process* is like a Poisson process, except that the jumps are of random size. Next, the definition of Compound Poisson processes, given in [24], is presented:

**Definition 2.1.3 (Compound Poisson process).** Let  $N_t$  be a Poisson process with intensity  $\lambda$  and let  $Y_1, Y_2, \dots$  be a sequence of identically distributed random variables with mean  $\beta = \mathbb{E}(Y_i)$ . We assume the random variables  $Y_1, Y_2, \dots$  are independent of



one another and also independent of the Poisson process  $N_t$ . We define the compound Poisson process as

$$Q_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \quad (2.2)$$

The Poisson process and the Wiener process are fundamental examples of the Lévy processes, named in honor of the French mathematician Paul Lévy. Next, the formal definition of a Lévy process is given:

**Definition 2.1.4 (Lévy process).** *A stochastic process  $(X_t)_{t \geq 0}$  on a filtered probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$  with values in  $\mathbb{R}$  such that  $X_0 = 0$  is called a Lévy process if it possesses the following properties:*

1. *Independent increments: for every increasing sequence of times  $t_0 \dots t_n$ , the random variables  $X_{t_0}$ ,  $X_{t_1} - X_{t_0}$ , ...,  $X_{t_n} - X_{t_{n-1}}$  are independent.*
2. *Stationary increments: the law of  $X_{t+h} - X_t$  does not depend on  $t$ ;*
3. *Stochastic continuity:  $\forall \epsilon > 0, \lim_{h \rightarrow 0} (\mathbb{P}(|X_{t+h} - X_t| \geq \epsilon)) = 0$ .*

The third condition does not imply in any way that the *sample paths* are continuous: as noted in Proposition 2.1.1, it is verified by the Poisson process. It means that, for a given time  $t$ , the probability of seeing a jump at  $t$  is zero: discontinuities occur at random times.

If we sample a Lévy process at regular times intervals  $0, \Delta, 2\Delta, \dots$ , we obtain a random walk: defining  $S_n(\Delta) \equiv X_{n\Delta}$ , we can write  $S_n(\Delta) = \sum_{k=0}^{n-1} Y_k$  where  $Y_k = X_{(k+1)\Delta} - X_{k\Delta}$  are i.i.d. random variables whose distribution is the same as the distribution of  $X_\Delta$ .

Choosing  $t = n\Delta$  with  $n = 0, 1, \dots$ , we see that for any  $t > 0$  and any  $n \geq 1$ ,  $X_t = S_n(\Delta)$  can be represented as a sum of  $n$  i.i.d. random variables whose distribution is that of  $X_{t/n}$ :  $X_t$  can be “divided” into  $n$  i.i.d. parts. A distribution having this property is said to be infinitely divisible:

**Definition 2.1.5 (Infinite divisibility).** *A probability distribution  $F$  on  $\mathbb{R}$  is said to be infinitely divisible if for any integer  $n \geq 2$ , there exists  $n$  i.i.d. random variables  $Y_1, \dots, Y_n$  such that  $Y_1 + \dots + Y_n$  has distribution  $F$ .*

Note that, if  $X$  is a Lévy process, for any  $t > 0$  the distribution of  $X_t$  is infinitely divisible.

Next, we present the definition of the characteristic function, based on [23].

**Definition 2.1.6 (Characteristic function).** *The characteristic function  $\phi$  of a distribution, or equivalently of a random variable  $X$ , is the Fourier transform of the distribution function  $F(x) = P(X \leq x)$ :*

$$\phi_X(\omega) = \mathbf{E}[e^{i\omega X}] = \int_{-\infty}^{+\infty} e^{i\omega x} dF(x). \quad (2.3)$$

The characteristic function has the following properties:

- $\phi(0) = 1$  and  $|\phi(\omega)| \leq 1$  for all  $\omega \in \mathbb{R}$ ;
- The characteristic function always exists and is continuous;
- $\phi$  determines the distribution function  $F$  uniquely, that is, random variables with the same characteristic function are identically distributed;
- It is possible to derive the moments of the random variable from  $\phi$ .

Knowing the characteristic function is essential to the study of a stochastic process due to the fact that we often do not know the distribution function of such a process in closed-form, while the characteristic function is explicitly known. Moreover, knowing the characteristic function plays a major role in the COS method being studied in this work. With particular regard to the Lévy processes, the characteristic function is given by:

**Proposition 2.1.2 (Characteristic function of a Lévy process).** *Let  $(X_t)_{t \geq 0}$  be a Lévy process on  $\mathbb{R}$ . There exists a continuous function  $\psi$  called the characteristic exponent of  $X$ , such that:*

$$\mathbf{E}[e^{i\omega X_t}] = e^{t\psi(\omega)}, \quad \omega \in \mathbb{R} \quad (2.4)$$

*Proof:* Please see [8].

The Lévy-Khintchine representation, presented as follows, gives us a closed form for the function  $\psi$ :

**Theorem 2.1.2 (Lévy-Khintchine representation).** *Let  $(X_t)_{t \geq 0}$  be a Lévy process on  $\mathbb{R}$  associated with a triplet  $(\mu; \sigma; \nu)$ , where  $\mu \in \mathbb{R}$ ;  $\sigma \in \mathbb{R}_0^+$  and  $\nu$  is a positive measure on  $\mathbb{R} \setminus \{0\}$ , not necessarily finite.*

*Then*

$$\mathbf{E}[e^{i\omega X_t}] = e^{t\psi(\omega)}, \quad \omega \in \mathbb{R}$$

*with*

$$\psi(\omega) = i\mu\omega - \frac{1}{2}\sigma^2\omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx). \quad (2.5)$$

*where the measure  $\nu$ , called the Lévy measure of  $X$ , must satisfy:*

$$\int_{\mathbb{R}} \inf\{1, x^2\} \nu(dx) = \int_{\mathbb{R}} (\{1 \wedge x^2\}) \nu(dx) < \infty$$

*Proof:* Please see [8].

From equation (2.5) one may conclude that, in the most general case, a Lévy process

consists of three independent parts or fundamental processes: a linear deterministic part, where  $\mu$  is called the *drift term*, a Brownian part with a *diffusion* coefficient  $\sigma$ , and a *pure jump* process whose dynamics is dictated by the Lévy measure  $\nu(dx)$ . The measure  $\nu(dx)$  defines how the jumps happen, which occur according to a Poisson process with intensity  $\lambda = \int_{\mathbb{R}} \nu(dx)$ .

The standard is not to model the stock price process directly as a Lévy process, but as an exponential of a Lévy process. This ensures that the log return is also positive with independent and stationary increments. In the *exponential Lévy models*, the risk-neutral dynamics of  $S_t$  under  $\mathbb{Q}$  is represented as the exponential of a Lévy process:

$$S_t = S_0 e^{rt + X_t} \quad (2.6)$$

where  $X_t$  is a Lévy process (under  $\mathbb{Q}$ ).

## 2.2 Geometric Brownian Motion Process

The arithmetic Brownian motion, first proposed by Bachelier [1] can take on negative values. To correct this, Samuelson [22] introduced the geometric Brownian motion (GBM), with the property that every dollar of market value is subject to the same multiplicative or percentage fluctuations per unit time regardless of the absolute price of the stock.

The Black-Scholes model assumes that the price of an option on an asset is modelled by the GBM. Schoutens [23] described the Black-Scholes model as follows:

The time evolution of a stock price  $S = \{S_t, t \geq 0\}$  is modelled as follows. Consider how  $S$  will change in some small time interval from the present time  $t$  to a time  $t + \Delta t$  in the near future. Writing  $\Delta S_t$  for the change  $S_{t+\Delta t} - S_t$ , the return in this interval is  $\Delta S_t / S_t$ . It is economically reasonable to expect this return to decompose into two components, a *systematic* and a *random* part.

Let us first look at the systematic part. We assume that the expected return of the stock over a period of time is proportional to the length of the period considered. This means that in a short time interval  $[S_t, S_{t+\Delta t}]$  of length  $\Delta t$ , the expected increase in  $S$  is given by  $\mu S_t \Delta t$ , where  $\mu$  is some parameter representing the mean rate of the return of the stock. In other words, the deterministic part of the stock return is modelled by  $\mu \Delta t$ .

A stock price fluctuates stochastically, and a reasonable assumption is that the variance of the return over the time interval  $[S_t, S_{t+\Delta t}]$  is proportional to the length of the interval. So, the random part of the return is modelled by  $\sigma \Delta W_t$ , where  $\Delta W_t$  represents the (normally distributed) noise term (with variance  $\Delta t$ ) driving the stock-price dynamics, and  $\sigma > 0$  is the parameter that describes how much effect the noise

has – how much the stock price fluctuates. In total, the variance of the return equals  $\sigma^2 \Delta t$ . Thus  $\sigma$  governs how volatile the price is, and is called the *volatility* of the stock. Putting this together, we have

$$\Delta S_t = S_t(\mu \Delta t + \sigma \Delta W_t), \quad S_0 > 0.$$

In the limit, as  $\Delta t \rightarrow 0$ , we have the stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 > 0. \quad (2.7)$$

The above stochastic differential equation has the unique solution (please see, for example, [4]):

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \quad (2.8)$$

This exponential functional of Brownian motion is called geometric Brownian motion (GBM). Note that

$$\log S_t - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \quad (2.9)$$

has a normal distribution with mean  $(\mu - \frac{1}{2}\sigma^2)t$  and variance  $\sigma^2 t$ . Thus,  $S_t$  has a lognormal distribution.

Redefining the GBM processes as the following process  $X_t$ :

$$X_t = \mu t + \sigma W_t \quad (2.10)$$

where the drift  $\mu$  has no relation to the ones in equations (2.7) and (2.8). Its characteristic function can be obtained by directly using the definition of a characteristic function given in (2.3):

$$\begin{aligned} \phi_{X_t}(\omega) &= \int_{-\infty}^{+\infty} e^{i\omega x} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right) dx \\ &= \exp\left(i\mu\omega t - \frac{\sigma^2\omega^2}{2}t\right). \end{aligned} \quad (2.11)$$

## 2.3 The Variance Gamma Process

As described by Madan *et al.* [18], the variance gamma (VG) process is a three parameter generalization of the Brownian motion as a model for the dynamics of the logarithm of the stock price. This process is obtained by evaluating a Brownian

motion, with constant drift and volatility, at a *random time change* given by a gamma process. Each unit of calendar time may be viewed as having an economically relevant time length given by an independent random variable that has a gamma density with unit mean and positive variance. Under the VG process, the unit period continuously compounded return is normally distributed, conditional on the realization of a random time. This random time has a gamma density. The resulting stochastic process and associated option pricing model provide us with a robust three parameter model. In addition to the volatility of the Brownian motion, there are parameters that control for:

- (i) kurtosis, a symmetric increase in the left and right tail probabilities of the return distribution;
- (ii) skewness, that allows for asymmetry of the left and right tails of the return density.

An additional attractive feature of the model is that it nests the lognormal density and the Black-Scholes formula as a parametric special case.

Contrary to much of the literature on option pricing, the VG process for log stock prices has no continuous martingale component. In contrast, it is a pure jump process that accounts for high activity (as in the Brownian motion) by having an infinite number of jumps in any interval of time. The importance of introducing a jump component in modelling stock price dynamics has recently been noted in [3], who argue that pure diffusion based models have difficulties in explaining smile effects, particularly in, short-dated option prices.

### 2.3.1 VG as Brownian Motion with a Drift

Consider a Brownian motion with constant drift  $\theta$  and volatility  $\sigma$  given by

$$b(t; \theta, \sigma) = \theta t + \sigma W_t \quad (2.12)$$

where  $W_t$  is the standard Brownian motion. The gamma process  $\gamma(t; \mu, \nu)$ , with mean rate  $\mu$  and variance rate  $\nu$ , is the process of independent gamma increments over non-overlapping time intervals  $(t, t + h)$ . The density  $f_h(g)$  of the increment  $g = \gamma(t + h; \mu, \nu) - \gamma(t; \mu, \nu)$  is given by the gamma density function with mean  $\mu h$  and variance  $\nu h$ . Specifically

$$f_h(g) = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}} \frac{g^{\frac{\mu^2 h}{\nu} - 1} \exp\left(-\frac{\mu}{\nu} g\right)}{\Gamma\left(\frac{\mu^2 h}{\nu}\right)}, g > 0, \quad (2.13)$$

where  $\Gamma(x)$  is the gamma function. The gamma density has a characteristic function,  $\phi_\gamma(t)(u) = E[\exp(iu\gamma(t; \mu, \nu))]$ , given by

$$\phi_{\gamma(t)}(u) = \left( \frac{1}{1 - iu\frac{\nu}{\mu}} \right)^{\frac{\mu^2 t}{\nu}} \quad (2.14)$$

The dynamics of the continuous time gamma process is best explained by describing a simulation of the process. As the process is an infinitely divisible one, of independent and identically distributed increments over non-overlapping intervals of equal length, the simulation may be described in terms of the Lévy measure ([20]),  $k_\gamma(x)dx$  explicitly given by

$$k_\gamma(x)dx = \frac{\mu^2 \exp(-\frac{\mu}{\nu}x)}{\nu x} dx, \quad \text{for } x > 0 \quad \text{and} \quad 0 \quad \text{otherwise.} \quad (2.15)$$

The VG process  $X(t; \sigma, \nu, \theta)$  is defined in terms of the Brownian motion with drift  $b(t; \theta, \sigma)$  and the gamma process with unit mean rate,  $\gamma(t; 1, \nu)$ , as

$$X(t; \sigma, \nu, \theta) = b(\gamma(t; 1, \nu); \theta, \sigma). \quad (2.16)$$

The VG process is obtained by evaluating the Brownian motion at a time given by the gamma process. The VG process has three parameters:

- (i)  $\sigma$  the volatility of the Brownian motion;
- (ii)  $\nu$  the variance rate of the gamma time change;
- (iii)  $\theta$  the drift in the Brownian motion with drift.

The process therefore provides two dimensions of control on the distribution over and above that of the volatility. It is observed that control is attained over the skew via  $\theta$  and over kurtosis with  $\nu$ .

The density function for the VG process at time  $t$  can be expressed, conditional on the realization of the gamma time change  $g$  as a normal density function. The unconditional density may then be obtained by integrating out  $g$  and employing the density (2.13) for the time change  $g$ . This gives us the density for  $X(t)$ ,  $f_{X(t)}(X)$  as

$$f_{X(t)}(X) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(X - \theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\nu}-1} \exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} dg. \quad (2.17)$$

The characteristic function for the VG process,  $\phi_{X(t)}(\omega) = E[e^{i\omega X(t)}]$ , is

$$\phi_{X(t)}(\omega) = \left( \frac{1}{1 - i\theta\nu\omega + (\sigma^2\nu/2)\omega^2} \right)^{t/\nu}. \quad (2.18)$$

### 2.3.2 VG as a Difference of Gamma Processes

The VG process may also be expressed as the difference of two independent increasing gamma processes, with the following expression:

$$X(t; \sigma, \nu, \theta) = \gamma_p(t; \mu_p, \nu_p) - \gamma_n(t; \mu_n, \nu_n). \quad (2.19)$$

The explicit relation between the parameters of the gamma processes differenced in (2.19) and the original parameters of the VG process (2.16) is given by

$$\mu_p = \frac{1}{2} \sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2} \quad (2.20)$$

$$\mu_n = \mu_p - \theta \quad (2.21)$$

$$\nu_p = \mu_p^2 \nu \quad (2.22)$$

$$\nu_n = \mu_n^2 \nu \quad (2.23)$$

The Lévy measure for the VG process has three representations, two in terms of the parameterizations introduced above, as time changed the Brownian motion and the difference of two gamma processes, and the third in terms of a symmetric VG process subjected to a measure change induced by a constant relative risk aversion utility function as in [16].

When viewed as the difference of two gamma processes, as in (2.19), we may write the Lévy measure for  $X(t)$ , employing (2.15) as

$$k_X(x)dx = \begin{cases} \frac{\mu_n^2}{\nu_n} \frac{\exp\left(-\frac{\mu_n}{\nu_n}|x|\right)}{|x|} dx & \text{for } x < 0, \\ \frac{\mu_p^2}{\nu_p} \frac{\exp\left(-\frac{\mu_p}{\nu_p}x\right)}{x} dx & \text{for } x > 0. \end{cases} \quad (2.24)$$

We observe from (2.24) that the VG process inherits the property of an infinite arrival rate of price jumps from the gamma process. The role of the original parameters is more easily observed when we write the Lévy measure directly in terms of these parameters. In terms of  $(\sigma, \nu, \theta)$ , one may write the Lévy measure as

$$k_X(x)dx = \frac{\exp(\theta x/\sigma^2)}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right) dx \quad (2.25)$$

The special case of  $\theta = 0$  in (2.25) yields a Lévy measure that is symmetric about zero. This yields the symmetric VG process employed by Madan and Seneta [17] and Madan and Milne [16] for describing the statistical process of continuously compounded

returns. We also observe from (2.25) that, when  $\theta < 0$ , negative values of  $x$  receive a higher relative probability than the corresponding positive value. Hence, negative values of  $\theta$  give rise to a negative skewness. We note further that large values of  $\nu$  lower the exponential decay rate of the Lévy measure symmetrically around zero, and hence raise the likelihood of large jumps, thereby raising tail probabilities and kurtosis.

## 2.4 Heston Model

The Heston (1993) model assumes that the underlying price process  $(S_t)$  follows the diffusion

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_{1,t}, \quad (2.26)$$

where  $W_{1,t}$  is a Wiener process and the volatility follows an Ornstein-Uhlenbeck process

$$d\sqrt{v_t} = -\beta\sqrt{v_t}dt + \delta dW_{2,t}. \quad (2.27)$$

Applying Itô's lemma, we conclude that the variance  $v_t$  follows the process

$$dv_t = (\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t}dW_{2,t}. \quad (2.28)$$

Defining  $\kappa = 2\beta$ ,  $\theta = \delta^2/(2\beta)$ , and  $\sigma = 2\delta$ , (2.28) can be written as the familiar square-root process (used by Cox, Ingersoll, and Ross [9])

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t} \quad (2.29)$$

Combining (2.26) and (2.29) becomes that the Heston (1993) model is represented, as Rouah [21], by the bivariate system of stochastic differential equations (SDEs)

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t} \end{aligned} \quad (2.30)$$

where  $E^{\mathbb{P}}[dW_{1,t}dW_{2,t}] = \rho dt$ .

The parameters of the model are:

- $\mu$  the drift of the process for the stock;
- $\kappa > 0$  the mean reversion speed for the variance;
- $\theta > 0$  the mean reversion level for the variance;
- $\sigma > 0$  the volatility of the variance;
- $v_0 > 0$  the initial (time zero) level of the variance;
- $\rho \in [-1, 1]$  the correlation between the two Brownian motions,  $W_{1,t}$  and  $W_{2,t}$ .



If the condition

$$2\kappa\theta \geq \sigma^2 \quad (2.31)$$

holds, then the process never hits zero. This condition is known as the *Feller condition*. The stock price and variance follow the process in Equation (2.30) under the historical measure  $\mathbb{P}$ , also called the physical measure. For pricing purposes, however, we need the processes for  $(S_t, v_t)$  under the risk-neutral measure  $\mathbb{Q}$ . In the Heston model, this is done by modifying each SDE in Equation (2.30) separately by an application of Girsanov's theorem. The risk-neutral process for the stock price is

$$dS_t = rS_t dt + \sqrt{v_t}S_t d\tilde{W}_{1,t} \quad (2.32)$$

where

$$\tilde{W}_{1,t} = \left( W_{1,t} + \frac{\mu - r}{\sqrt{v_t}} t \right).$$

It is sometimes convenient to express the price process in terms of the log price instead of the price itself. By an application of Itô's lemma, the log price process is

$$d \ln S_t = \left( \mu - \frac{1}{2} \right) dt + \sqrt{v_t} dW_{1,t}.$$

The risk-neutral process for the log price is

$$d \ln S_t = \left( r - \frac{1}{2} \right) dt + \sqrt{v_t} d\tilde{W}_{1,t}. \quad (2.33)$$

If the stock pays a continuous dividend yield,  $q$ , then in equations (2.32) and (2.33) we replace  $r$  by  $r - q$ .

The risk-neutral process for the variance is obtained by introducing a function  $\lambda(S_t, v_t, t)$  into the drift of  $dv_t$  in Equation (2.30), as follows

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)] dt + \sigma\sqrt{v_t} d\tilde{W}_{2,t} \quad (2.34)$$

where

$$\tilde{W}_{2,t} = \left( W_{2,t} + \frac{\lambda(S_t, v_t, t)}{\sigma\sqrt{v_t}} t \right).$$

The function  $\lambda(S, v, t)$  is called the *volatility risk premium*. As explained in Heston [11], Breeden's [6] consumption model yields a premium proportional to the variance, so that  $\lambda(S, v, t) = \lambda v_t$ , where  $\lambda$  is a constant. Substituting for  $\lambda v_t$  in Equation (2.34), the risk-neutral version of the variance process is

$$dv_t = \kappa^*(\theta^* - v_t) dt + \sigma\sqrt{v_t} d\tilde{W}_{2,t} \quad (2.35)$$

where  $\kappa^* = \kappa + \lambda$  and  $\theta^* = \kappa\theta/(\kappa + \lambda)$  are the risk-neutral parameters of the variance process.

To summarize, the risk-neutral process is

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t}S_t d\tilde{W}_{1,t} \\ dv_t &= \kappa^*(\theta^* - v_t) dt + \sigma\sqrt{v_t} d\tilde{W}_{2,t} \end{aligned} \quad (2.36)$$

where  $E^{\mathbb{Q}}[d\tilde{W}_{1,t} d\tilde{W}_{2,t}] = \rho dt$  and with  $\mathbb{Q}$  the risk-neutral measure.

Note that, when  $\lambda = 0$ , we have  $\kappa^* = \kappa$  and  $\theta^* = \theta$  so that these parameters under the physical and risk-neutral measures are the same.

Standard arbitrage arguments ([5], [19]) demonstrate that the value of any asset  $U(S, v, t)$  (including accrued payments) must satisfy the partial differential equation (PDE):

$$\begin{aligned} \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} \\ + [\kappa(\theta - v) - \lambda(S, v, t)]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0. \end{aligned} \quad (2.37)$$

where  $\lambda(S, v, t)$  is the volatility risk premium defined above.

A European call option with maturity at time  $T$  and strike  $K$  satisfies the PDE (2.37) subject to the following boundary conditions:

$$\begin{aligned} U(S, v, T) &= \max(0, S - K), \\ U(0, v, t) &= 0; \\ \frac{\partial U}{\partial S}(\infty, v, t) &= 1, \\ rs\frac{\partial U}{\partial S}(S, 0, t) + \kappa\theta\frac{\partial U}{\partial v}(S, 0, t) - rU(S, 0, t) + U(S, 0, t) &= 0, \\ U(S, \infty, t) &= S. \end{aligned} \quad (2.38)$$

We can define the log price  $x = \ln S$  and express the PDE (2.37) in terms of  $(x, v, t)$  instead of  $(S, v, t)$ . Then, as demonstrated in [21], all the  $S$  terms are canceled, and we obtain the Heston PDE in terms of the log price  $x = \ln S$

$$\begin{aligned} \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \rho\sigma v\frac{\partial^2 U}{\partial v\partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \left(r - \frac{1}{2}v\right)\frac{\partial U}{\partial x} \\ + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0. \end{aligned} \quad (2.39)$$

where we have substituted  $\lambda(S, v, t) = \lambda v$ .

Recall equation (1.6) for the European call price, written here using  $x = x_t = \ln S_t$

$$v^{call}(x, K, T) = e^x P_1 - K e^{-r\tau} P_2. \quad (2.40)$$

Rouah [21] shows that also  $P_1$  and  $P_2$  satisfy the Heston PDE:

$$\begin{aligned} \frac{1}{2}v \frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v \frac{\partial^2 P_j}{\partial v \partial x} + \frac{1}{2}\sigma^2 v \frac{\partial^2 P_j}{\partial v^2} + (r + u_j v) \frac{\partial P_j}{\partial x} \\ + (a - b_j v) \frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0. \end{aligned} \quad (2.41)$$

for  $j = 1, 2$  and where  $u_1 = \frac{1}{2}$ ,  $u_2 = -\frac{1}{2}$ ,  $a = \kappa\theta$ ,  $b_1 = \kappa + \lambda - \rho\sigma$  and  $b_2 = \kappa + \lambda$ .

Heston [11] postulates that the characteristic functions for the logarithm of the terminal stock price,  $x_T = \ln S_T$ , are of the log linear form

$$f_j(\omega; x_t, v_t) = \exp[C_j(\tau, \omega) + D_j(\tau, \omega)v_t + i\omega x_t] \quad (2.42)$$

where  $C_j(\tau, \omega)$  and  $D_j(\tau, \omega)$  are given by

$$C_j(\tau, \omega) = r\omega\tau i + \frac{\kappa\theta}{\sigma^2} \left[ (\beta_j + d_j)\tau - 2 \ln \left( 1 - \frac{g_j e^{d_j \tau}}{1 - g_j} \right) \right], \quad (2.43)$$

$$D_j(\tau, \omega) = \frac{\beta_j + d_j}{\sigma^2} \left( \frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}} \right), \quad (2.44)$$

with

$$d_j = \sqrt{\beta_j^2 - 4\hat{\alpha}_j\gamma}, \quad g_j = \frac{\beta_j + d_j}{\beta_j - d_j} \quad (2.45)$$

and auxiliary variables

$$\hat{\alpha}_j = u_j \omega i - \frac{1}{2}\omega^2, \quad \beta_j = b_j - \rho\sigma\omega i, \quad \gamma = \frac{1}{2}\sigma^2. \quad (2.46)$$

It makes sense that two characteristic functions  $f_1$  and  $f_2$  be associated with the Heston model, because  $P_1$  and  $P_2$  are obtained under different measures. On the other hand it also seems that only a single characteristic function ought to exist, because there is only one underlying stock price in the model. Rouah [21] shows that the “true” characteristic function is actually  $f_2$ . So, the characterisitic function of  $x_T = \ln S_T$  now reads

$$\phi(\omega; x_t, v_t) = \exp[C(\tau, \omega) + D(\tau, \omega)v_t + i\omega x_t] \quad (2.47)$$

where  $C(\tau, \omega)$  and  $D(\tau, \omega)$  are given by

$$C(\tau, \omega) = r\omega\tau i + \frac{\kappa\theta}{\sigma^2} \left[ (\beta + d)\tau - 2 \ln \left( 1 - \frac{g e^{d\tau}}{1 - g} \right) \right], \quad (2.48)$$

$$D(\tau, \omega) = \frac{\beta + d}{\sigma^2} \left( \frac{1 - e^{d\tau}}{1 - g e^{d\tau}} \right), \quad (2.49)$$

with

$$d = \sqrt{\beta^2 - 4\hat{\alpha}\gamma}, \quad g = \frac{\beta + d}{\beta - d} \quad (2.50)$$

and auxiliary variables

$$\hat{\alpha} = -\frac{1}{2}\omega(i + \omega), \quad \beta = \kappa - \rho\sigma\omega i, \quad \gamma = \frac{1}{2}\sigma^2. \quad (2.51)$$

## Chapter 3

# The COS Method

The present chapter describes the details of the COS method. In Section 3.1, we introduce the Fourier-cosine expansion for solving inverse Fourier integrals. Based on this, we derive, in Section 3.2, the formulas for pricing European options. We focus on the Lévy and the Heston processes for the underlying.

### 3.1 Fourier Integrals and Cosine Series

The point of departure for pricing European options with numerical integration techniques is the risk-neutral formula:

$$v(x, t_0) = e^{-r\tau} \mathbb{E}^{\mathbb{Q}}[v(y, T)|x] = e^{-r\tau} \int_{\mathbb{R}} v(y, T) f(y|x) dy, \quad (3.1)$$

where  $v$  denotes the option value,  $\tau$  is the difference between the maturity,  $T$ , and the initial date,  $t_0$ ,  $\mathbb{E}^{\mathbb{Q}}[\cdot]$  is the expectation operator under the risk-neutral measure  $\mathbb{Q}$ ,  $x$  and  $y$  are state variables at times  $t_0$  and  $T$ , respectively;  $f(y|x)$  is the probability density of  $y$  given  $x$ , and  $r$  is the risk-neutral interest rate.

The density and its characteristic function,  $f(x)$  and  $\phi(\omega)$ , form an example of a Fourier pair,

$$\phi(\omega) = \int_{\mathbb{R}} e^{ix\omega} f(x) dx, \quad (3.2)$$

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\omega} \phi(\omega) d\omega. \quad (3.3)$$

#### 3.1.1 Inverse Fourier Integral via Cosine Expansion

In this section, as a first step, we present a different methodology for solving, in particular, the inverse Fourier integral in (3.3). The main idea is to reconstruct the whole integral - not just the integrand - from its Fourier-cosine series expansion (also

called “cosine expansion”), extracting the series coefficients directly from the integrand. For a function supported on  $[0, \pi]$ , the cosine expansion reads

$$f(\theta) = \frac{A_0}{2} + \sum_{k=1}^{+\infty} A_k \cdot \cos(k\theta) \quad \text{with} \quad A_k = \frac{2}{\pi} \int_0^\pi f(\theta) \cos(k\theta) d\theta, \quad (3.4)$$

For functions supported on any other finite interval, say  $[a, b] \in \mathbb{R}$ , the Fourier-cosine series expansion can easily be obtained via a change of variables:

$$\theta := \frac{x-a}{b-a}\pi, \quad x = \frac{b-a}{\pi}\theta + a.$$

It then reads

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{+\infty} A_k \cdot \cos\left(k\pi \frac{x-a}{b-a}\right), \quad (3.5)$$

with

$$A_k = \frac{2}{b-a} \int_a^b f(x) \cos\left(k\pi \frac{x-a}{b-a}\right) dx. \quad (3.6)$$

Since any real function has a cosine expansion when it is finitely supported, the derivation starts with a truncation of the infinite integration range in (3.3). Due to the conditions for the existence of a Fourier transform, the integrands in (3.3) have to decay to zero at  $\pm\infty$  and we can truncate the integration range in a proper way without losing accuracy.

Suppose  $[a, b] \in \mathbb{R}$  is chosen such that the truncated integral approximates the infinite counterpart very well, i.e.,

$$\phi_1(\omega) := \int_a^b e^{i\omega x} f(x) dx \approx \int_{\mathbb{R}} e^{i\omega x} f(x) dx = \phi(\omega). \quad (3.7)$$

Using Euler’s Formula, we can rewrite (3.6) as

$$A_k = \frac{2}{b-a} \operatorname{Re} \left\{ \int_a^b e^{i(k\pi \frac{x-a}{b-a})} f(x) dx \right\}. \quad (3.8)$$

where  $\operatorname{Re}\{\cdot\}$  denotes taking the real part of the argument. Noting that

$$k\pi \frac{x-a}{b-a} = \frac{k\pi}{b-a}x - \frac{ak\pi}{b-a}$$

we find that

$$A_k = \frac{2}{b-a} \operatorname{Re} \left\{ \int_a^b e^{i \frac{k\pi}{b-a}x} f(x) dx \cdot e^{-i \frac{ak\pi}{b-a}} \right\}. \quad (3.9)$$

Combining (3.9) with (3.7), the  $A_k$  coefficients reads

$$A_k \equiv \frac{2}{b-a} \operatorname{Re} \left\{ \phi_1 \left( \frac{k\pi}{b-a} \right) \cdot \exp \left( -i \frac{ak\pi}{b-a} \right) \right\}, \quad (3.10)$$

It then follows from (3.7) that  $A_k \approx F_k$  with

$$F_k \equiv \frac{2}{b-a} \operatorname{Re} \left\{ \phi \left( \frac{k\pi}{b-a} \right) \cdot \exp \left( -i \frac{ak\pi}{b-a} \right) \right\}. \quad (3.11)$$

We now replace  $A_k$  by  $F_k$  in the series expansion of  $f(x)$  on  $[a, b]$ , i.e.,

$$f_1(x) = \frac{F_0}{2} + \sum_{k=1}^{+\infty} F_k \cdot \cos \left( k\pi \frac{x-a}{b-a} \right), \quad (3.12)$$

and truncate the series summation such that

$$f_2(x) = \frac{F_0}{2} + \sum_{k=1}^{N-1} F_k \cdot \cos \left( k\pi \frac{x-a}{b-a} \right). \quad (3.13)$$

The resulting error in  $f_2(x)$  consists of two parts: a series truncation error from (3.12) to (3.13) and an error originating from the approximation of  $A_k$  by  $F_k$ .

## 3.2 Pricing European Options

In this section, the COS formula is derived for European-style options by replacing the density function by its Fourier-cosine series. We make use of the fact that a density function tends to be smooth and, therefore, only a few terms in the expansion may already give a good approximation.

Since the density rapidly decays to zero as  $y \rightarrow \pm\infty$  in (3.1), we truncate the infinite integration range without losing significant accuracy to  $[a, b] \subset \mathbb{R}$ , and we obtain approximation  $v_1$ :

$$v_1(x, t_0) = e^{-r\tau} \int_a^b v(y, T) f(y|x) dy. \quad (3.14)$$

We will give insight into the choice of  $[a, b]$  in Chapter 4.

In the second step, since  $f(y|x)$  is usually not known whereas the characteristic function is, we replace the density by its cosine expansion in  $y$ ,

$$f(y|x) = \frac{A_0(x)}{2} + \sum_{k=1}^{+\infty} A_k(x) \cos(k\pi \frac{y-a}{b-a}), \quad (3.15)$$

with

$$A_k(x) := \frac{2}{b-a} \int_a^b f(y|x) \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \quad (3.16)$$

so that

$$v_1(x, t_0) = e^{-r\tau} \int_a^b v(y, T) \left( \frac{A_0(x)}{2} + \sum_{k=1}^{+\infty} A_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right) \right) dy. \quad (3.17)$$

We interchange the summation and integration, and insert the definition

$$V_k := \frac{2}{b-a} \int_a^b v(y, T) \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \quad (3.18)$$

resulting in

$$v_1(x, t_0) = \frac{1}{2}(b-a)e^{-r\tau} \left( \frac{A_0(x)V_0}{2} + \sum_{k=1}^{+\infty} A_k(x)V_k \right). \quad (3.19)$$

Note that the  $V_k$  are the cosine series coefficients of payoff function  $v(y, T)$  in  $y$ . Thus, from (3.14) to (3.19) we have transformed the product of two real functions,  $f(y|x)$  and  $v(y, T)$ , into that of their Fourier-cosine series coefficients.

Due to the rapid decay rate of these coefficients, we further truncate the series summation to obtain approximation  $v_2$ :

$$v_2(x, t_0) = \frac{1}{2}(b-a)e^{-r\tau} \left( \frac{A_0(x)V_0}{2} + \sum_{k=1}^{N-1} A_k(x)V_k \right). \quad (3.20)$$

Similar to Section 3.1, coefficients  $A_k(x)$  defined in (3.16) can be approximated by  $F_k(x)$  as defined in (3.11). Replacing  $A_k(x)$  in (3.20) by  $F_k(x)$ , we obtain

$$v(x, t_0) \approx v_3(x, t_0) = e^{-r\tau} \left( \frac{1}{2} \text{Re} \{ \phi(0; x) \} V_0 + \sum_{k=1}^{N-1} \text{Re} \left\{ \phi \left( \frac{k\pi}{b-a}; x \right) e^{-ik\pi \frac{a}{b-a}} \right\} V_k \right), \quad (3.21)$$

with characteristic function  $\phi$ . This is the COS formula for general underlying processes. In section 3.2.1, we will show that the  $V_k$  can be obtained analytically for plain vanilla options, and section 3.2.2 shows that (3.21) can be simplified for the Lévy and the Heston models, so that many strikes can be handled simultaneously.

The key step in obtaining this semi-analytic formula (3.21) for option pricing is the replacement of the probability density function by its Fourier-cosine series expansion. The advantage is that the product of the density and the payoff is transformed into a linear combination of products of cosine basis functions and a (payoff) function which is known analytically.

Important for convergence is therefore the convergence of the cosine series of the density function, not the cosine series of the payoff, which appears only because we



interchanged the summation and the integration in (3.19).

### 3.2.1 Coefficients $V_k$ for Plain Vanilla Options

Before we can use (3.21) for pricing options, the payoff series coefficients,  $V_k$ , have to be recovered. Let us assume that the characteristic function of the log-asset price is known and we represent the payoff as a function of the log-asset price. Thus, we denote the log-asset prices by

$$x := \ln(S_0/K) \quad \text{and} \quad y := \ln(S_T/K),$$

with  $S_t$  being the underlying price at time  $t$  and  $K$  the strike price. The payoff for European options, in log-asset price, reads

$$v(y, T) \equiv K[\alpha \cdot (e^y - 1)]^+ \quad \text{with} \quad \alpha = \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put.} \end{cases}$$

Before deriving  $V_k$  from its definition in (3.18), we need the next mathematical results:

**Theorem 3.2.1 (Coefficients  $V_k$ ).** *The cosine series coefficients,  $\chi_k$ , of  $g(y) = e^y$  on  $[c, d] \subset [a, b]$ ,*

$$\chi_k(c, d) := \int_c^d e^y \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \quad (3.22)$$

*and the cosine series coefficients,  $\psi_k$ , of  $g(y) = 1$  on  $[c, d] \subset [a, b]$ ,*

$$\psi_k(c, d) := \int_c^d \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \quad (3.23)$$

*are known analytically.*

*Proof.* Using integration by parts, we conclude that

$$\int e^y \cos\left(k\pi \frac{y-a}{b-a}\right) dy = \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2} \left[ e^y \cos\left(k\pi \frac{y-a}{b-a}\right) + \frac{k\pi}{b-a} e^y \sin\left(k\pi \frac{y-a}{b-a}\right) \right]$$

Thus,

$$\begin{aligned} \chi_k(c, d) = & \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2} \left[ e^d \cos\left(k\pi \frac{d-a}{b-a}\right) - e^c \cos\left(k\pi \frac{c-a}{b-a}\right) \right. \\ & \left. + \frac{k\pi}{b-a} e^d \sin\left(k\pi \frac{d-a}{b-a}\right) - \frac{k\pi}{b-a} e^c \sin\left(k\pi \frac{c-a}{b-a}\right) \right] \quad (3.24) \end{aligned}$$

With regard to  $\psi_k$ , we observe that

$$\int \cos\left(k\pi \frac{y-a}{b-a}\right) dy = \begin{cases} \frac{b-a}{k\pi} \sin\left(k\pi \frac{y-a}{b-a}\right) & \text{for } k \neq 0, \\ y & \text{for } k = 0. \end{cases}$$

Therefore,

$$\psi_k(c, d) = \begin{cases} \frac{b-a}{k\pi} \left[ \sin\left(k\pi \frac{d-a}{b-a}\right) - \sin\left(k\pi \frac{c-a}{b-a}\right) \right] & \text{for } k \neq 0, \\ d - c & \text{for } k = 0. \end{cases} \quad (3.25)$$

□

Now, focusing, for example, on a call option, we obtain

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a} \int_a^b K[(e^y - 1)]^+ \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a} K \int_0^b (e^y - 1) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a} K(\chi_k(0, b) - \psi_k(0, b)) \end{aligned} \quad (3.26)$$

where  $\chi_k$  and  $\psi_k$  are given by (3.24) and (3.25), respectively. Similarly, for a vanilla put, we find

$$V_k^{put} = \frac{2}{b-a} K(-\chi_k(a, 0) + \psi_k(a, 0)). \quad (3.27)$$

### 3.2.2 Formula for Exponential Lévy Processes and the Heston Model

Note that (3.21) is greatly simplified for the Lévy and the Heston models. Here we use boldfaced values to differentiate vectors.

For the Lévy processes, whose characteristic functions can be represented by

$$\phi(\omega; \mathbf{x}) = \varphi_{levy}(\omega) \cdot e^{i\omega \mathbf{x}} \quad \text{with} \quad \varphi_{levy}(\omega) := \phi(\omega; 0), \quad (3.28)$$

the pricing formula (3.21) is simplified to

$$v(\mathbf{x}, t_0) \approx e^{-r\tau} \left( \frac{1}{2} \text{Re} \{ \varphi_{levy}(0) \} \mathbf{V}_0 + \sum_{k=1}^{N-1} \text{Re} \left\{ \varphi_{levy} \left( \frac{k\pi}{b-a} \right) e^{ik\pi \frac{\mathbf{x}-a}{b-a}} \right\} \mathbf{V}_k \right), \quad (3.29)$$

Recalling the  $V_k$ -formulas for vanilla European options in (3.26) and (3.27), we can now present them as a vector multiplied by a scalar,

$$\mathbf{V}_k = U_k \mathbf{K},$$

where

$$U_k = \begin{cases} \frac{2}{b-a}(\chi_k(0, b) - \psi_k(0, b)) & \text{for a call,} \\ \frac{2}{b-a}(-\chi_k(a, 0) + \psi_k(a, 0)) & \text{for a put.} \end{cases} \quad (3.30)$$

As a result, the pricing formula reads

$$v(\mathbf{x}, t_0) \approx \mathbf{K} e^{-r\tau} \cdot \text{Re} \left\{ \frac{1}{2} \varphi_{levy}(0) U_0 + \sum_{k=1}^{N-1} \varphi_{levy} \left( \frac{k\pi}{b-a} \right) U_k \cdot e^{ik\pi \frac{\mathbf{x}-a}{b-a}} \right\}, \quad (3.31)$$

where the summation can be written as a matrix-vector product if  $\mathbf{K}$  (and therefore  $\mathbf{x}$ ) is a vector. Note that, as the  $U_k$  values are real, we can interchange  $\text{Re} \{ \cdot \}$  and  $\sum$ , which simplifies the implementation in MATLAB. It should be pointed out that equation (3.31) is an expression with independent variable  $\mathbf{x}$ . It is therefore possible to obtain the option prices for different strikes in one single numerical experiment by choosing a  $\mathbf{K}$ -vector as the input vector.

In chapter 2, we have already presented the characteristic functions of the GBM and the VG models. Now, using this approach we obtain for the GBM

$$\phi_{levy}(\omega) = \exp \left( i\mu\omega\tau - \frac{\sigma^2\omega^2}{2}\tau \right). \quad (3.32)$$

and for the VG model

$$\phi_{levy}(\omega) = \left( \frac{1}{1 - i\theta\nu\omega + (\sigma^2\nu/2)\omega^2} \right)^{\tau/\nu}. \quad (3.33)$$

For the Heston (1993) model, the COS pricing equation is also simplified, since

$$\phi(\omega; \mathbf{x}, v_0) = \varphi_{hes}(\omega; v_0) \cdot e^{i\omega\mathbf{x}}; \quad (3.34)$$

with  $v_0$  the volatility of the underlying at the initial time and  $\varphi_{hes}(\omega, v_0) := \phi(\omega; 0, v_0)$ . We then find

$$v(\mathbf{x}, t_0, u_0) \approx \mathbf{K} e^{-r\tau} \cdot \text{Re} \left\{ \frac{1}{2} \varphi_{hes}(0; v_0) U_0 + \sum_{k=1}^{N-1} \varphi_{hes} \left( \frac{k\pi}{b-a}; v_0 \right) U_k \cdot e^{ik\pi \frac{\mathbf{x}-a}{b-a}} \right\}, \quad (3.35)$$

Then, recalling equation (2.47)

$$\varphi_{hes}(\omega; v_0) = \exp[C(\tau, \omega) + D(\tau, \omega)v_0], \quad (3.36)$$

where  $C(\tau, \omega)$  and  $D(\tau, \omega)$  are defined in equations (2.48) to (2.51).

## Chapter 4

# Numerical Results

In this section, we perform some numerical tests to evaluate the efficiency and accuracy of the COS method. First, in Section 4.1, the setup of the numerical experiments is described. Then, in the following sections, we focus on plain vanilla European options and consider different processes for the underlying asset. In Section 4.2, we consider the GBM. Then, the infinite activity Lévy process VG is addressed in Section 4.3. Finally, the stochastic volatility process Heston (1993) is considered in Section 4.4.

### 4.1 Experimental Setup

The computer used for all experiments has an Intel Core2 Duo CPU processor, with 2.53 Ghz and 6.00 GB RAM; the code is written in MATLAB based on [14].

All CPU times are presented in milliseconds and are determined after averaging the computing times obtained from  $10^4$  experiments.

The parameter  $N$  in the experiments to follow denotes the number of terms in the Fourier cosine expansion.

To determine the interval  $[a, b]$  within the COS method, we propose the following:

$$[a, b] := \left[ c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}} \right] \quad \text{with } L = 10. \quad (4.1)$$

Here,  $c_n$  denotes the  $n$ -th cumulant of  $\ln(S_T/K)$ . The cumulants for the models employed are presented in Appendix A.

Note that, when pricing call options, the accuracy of the method exhibits some sensitivity regarding the choice of parameter  $L$  in (4.1). In fact, a call payoff grows exponentially with the log-stock price, which may introduce a significant cancellation error for large values of  $L$ . Put options do not suffer from this, as their payoff value is bounded by the value  $K$ . Thus, for pricing call options, one can therefore either stay with  $L \in [7.5, 10]$  or rely on the well-known put-call parity in (1.7), which is equivalent

to

$$v^{call}(\mathbf{x}, t_0) = v^{put}(\mathbf{x}, t_0) - Ke^{-r\tau} + S_t. \quad (4.2)$$

In the experiments that follows, we use (4.2) when pricing calls, which gives a slightly higher accuracy than directly applying (3.29) with (4.1).

To illustrate error convergence we have plotted the grid size  $2^n$  against the logarithmic absolute error given by

$$\log \text{absol error} = \log_{10}(|v^{call} - v_{ref}^{call}|)$$

## 4.2 GBM

The first set of call option experiments is performed under the GBM process with a short time to maturity. The parameters selected for this test are

$$S_0 = 100, \quad r = 0.1, \quad q = 0, \quad T = 0.1, \quad \sigma = 0.25 \quad (4.3)$$

As shown in Figure 4.1, the recovered density function with the small maturity time

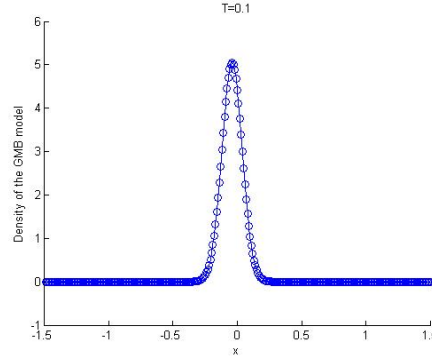


Figure 4.1: Recovered density function of the GBM model;  $K = 100$ , with other parameters as in (4.3)

$T = 0.1$  does not have fat tails.

Figure 4.2 shows that the error convergence of the COS method is exponential and that with  $N = 6$ , the COS results already coincide with the reference values. Furthermore, we observe that the error convergence rate is basically the same for the different strikes.

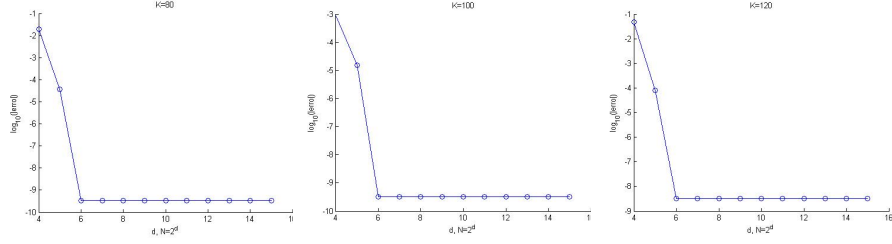


Figure 4.2: COS error convergence for pricing European call options under the GBM model.

In Table 4.1, we present information about the CPU time and error convergence for pricing European call options at  $K=80, 100$  and  $120$ . The maximum error of the option values over the three strike prices is presented. The results for these strikes are obtained in one single computation.

Table 4.1: Error convergence and CPU time using the COS method for European calls under GBM, with parameters as in (4.3);  $K=80, 100, 120$ ; reference values= $20.799226309 \dots, 3.659968453 \dots$  and  $0.04477814$ , respectively.

	N	16	32	64	128	256
COS	msec	1.968	1.972	2.033	2.301	2.564
	max.abs.error	4.72e-02	7.89e-05	3.33e-09	3.33e-09	3.33e-09

### 4.3 VG

As a second test, we evaluate the convergence of the method for calls under the VG model, which belongs to the class of infinite activity Lévy processes. The parameters selected in the numerical experiments are

$$K = 90, \quad S_0 = 100, \quad r = 0.1, \quad q = 0, \quad \sigma = 0.12, \quad \theta = -0.14, \quad \nu = 0.2, \quad L = 10. \quad (4.4)$$

Here we compare the convergence for  $T = 1$  year and for  $T = 0.1$  year.

Figure 4.3 presents the difference in shape of the two recovered density functions. For  $T = 0.1$ , the density is much more peaked than for  $T = 1$ . Results are summarized in Table 4.2. Note that for  $T = 0.1$ , the error convergence of the COS method is algebraic instead of exponential. This is in agreement with the recovered density function in Figure 4.3, which is clearly not differentiable in  $[a, b]$ .

We also plot the errors in Figure 4.4.

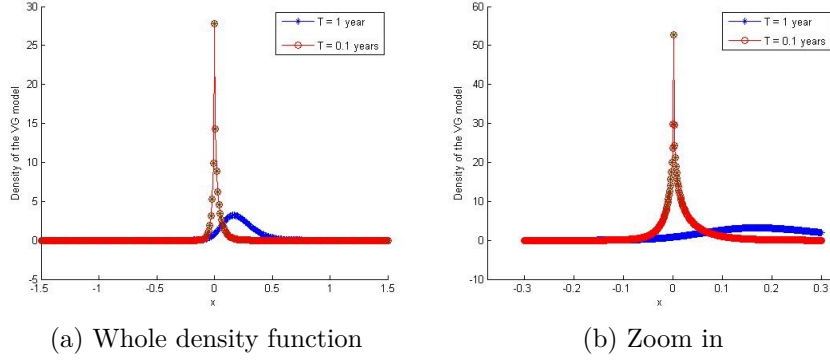


Figure 4.3: Recovered density functions for the VG model and two maturity dates, with parameters as in (4.4).

## 4.4 Heston

Finally, we have chosen the Heston model and price calls with the following parameters:

$$\begin{aligned}
 S_0 = 100, \quad K = 100, \quad r = 0, \quad q = 0, \quad \kappa = 1.5768, \quad \sigma = 0.5751 \\
 \theta = 0.0398 \quad v_0 = 0.0175, \quad \rho = -0.5711
 \end{aligned} \tag{4.5}$$

We consider two maturities,  $T = 1$  and  $T = 10$ . Since the analytic formula for  $c_4$  is involved, we define the truncation range, instead of (4.1), by

$$[a, b] := [c_1 - 12\sqrt{|c_2|}, c_1 + 12\sqrt{|c_2|}]$$

Cumulant  $c_2$  may become negative for sets of Heston parameters that do not satisfy the Feller condition in (2.31). We therefore use the absolute value of  $c_2$ . Figure 4.5 presents the recovered density functions of the Heston model for two maturities,  $T = 1$

Table 4.2: Convergence of the COS method for a call under the VG model with parameters as in (4.4).

COS method					
T=0.1; Reference values = 10.993703187			T = 1; Reference values = 19.099354724		
N	Error	Time (msec.)	N	Error	Time (msec.)
64	2.73e-04	1.91	32	2.3e-03	1.92
128	-3.13e-05	2.05	64	1.88e-05	1.93
256	1.15e-06	2.16	96	1.09e-08	1.96
512	5.05e-06	2.39	128	-1.28e-08	2.01
1024	-4.06e-07	2.87	160	-4.03e-10	2.06

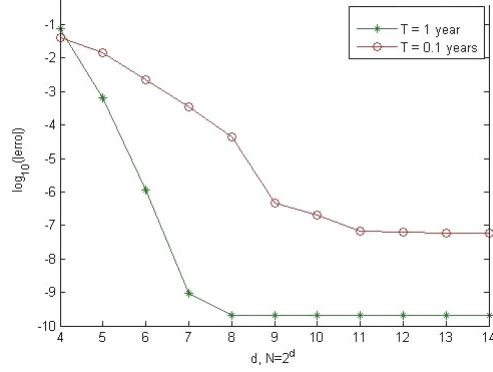


Figure 4.4: COS error convergence for pricing European call options under the VG model.

and  $T = 10$ . It shows that  $T = 1$  gives rise to a sharper-peaked density than  $T = 10$ . Tables 4.3 and 4.4 illustrate the high efficiency of the COS method.

The convergence rate of the COS method is slower for the short maturity example, as compared to the 10-year maturity. This is due to the fact that the density function for the latter case is smoother, as seen in Figure 4.5. However, the COS convergence rate for  $T = 1$  is still exponential.

Table 4.3: Error convergence and CPU times for the COS method for calls under the Heston model with  $T = 1$ , with parameters as in (4.5); reference value= 5.785155450.

N	64	96	128	160	192
Error	-4.05e-04	1.46e-06	4.46e-08	-2.86e-09	7.46e-09
CPU Time(msec.)	1.977	1.977	2.026	2.099	2.099

Table 4.4: Error convergence and CPU times for the COS method for calls under the Heston model with  $T = 10$ , with parameters as in (4.5); reference value= 22.318945791.

N	32	64	96	128	160
Error	-5.19e-03	1.64e-05	-7.70e-08	-1.01e-08	-9.99e-09
CPU Time (msec.)	2.02	2.08	2.16	2.17	2.21



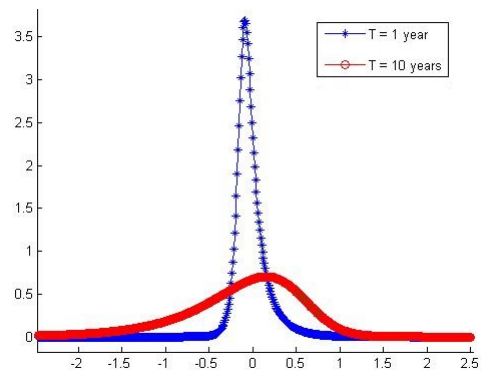


Figure 4.5: Recovered density function of the Heston model, with parameters as in (4.5).

## Chapter 5

# Conclusions

In this thesis, we have presented an option pricing method, based on the work of Fang and Oosterlee [10], for pricing European-style options and which is called, the COS method, based on Fourier-cosine series expansions. Similarly to other methods that are based on the knowledge of the characteristic function, it is flexible with respect to the choice of asset price process. This feature has been demonstrated in numerical examples for European options and for three driven processes, namely GBM, VG, and Heston. The key assumption of the method is that the series coefficients of many density functions can be accurately retrieved from their characteristic functions. As such, one can decompose a density function into a linear combination of cosine functions. It is this decomposition that makes the numerical computation of the risk-neutral valuation formula easy and highly efficient.

The main strengths of the method are its computational speed and accuracy. Regarding the numerical experiments, we were able to conclude that the convergence rate of the COS method is exponential, except when the density function of the underlying process has a discontinuity in one of its derivatives. In this case, an algebraic convergence is expected and has been observed.

Although the obtained running times were above the ones mentioned in Fang and Oosterlee [10], they are still very fast. Actually, for  $N < 150$ , all the CPU times are lower than 2.301 milliseconds.

With regard to future research, we would like to compare the COS method with other methods, namely the Carr-Madan method and the CONV method, in terms of speed and accuracy. Furthermore, since this thesis has only applied the COS method to valuating European-style options, it would be interesting to apply it as well to Bermudan and American Options.

## Appendix A

## Appendix A

### **Cumulants of $\ln(S_t/K)$**

The cumulants,  $c_n$  are defined by the cumulant generating function  $g(t)$ :

$$g(t) := \log(\mathbb{E}(e^{t \cdot X}))$$

for some random variable  $X$ . The  $n$ -th cumulant is given by the  $n$ -th derivative of  $g$  evaluated at  $t = 0$ . We present the cumulants  $c_1$ ,  $c_2$  and  $c_4$  needed to determine the truncation range in 4.1. For the price process discussed in this thesis, they are shown in Table A.1.

Table A.1: Cumulants for the GBM, VG and Heston models; and  $w$ , the drift correction term, which satisfies  $\exp(-wt) = \phi(-i, t)$ .

GBM	$c_1 = \mu T$ $c_2 = \sigma^2 T$ $c_4 = 0$ $w = 0$
VG	$c_1 = (\mu + \theta)T$ $c_2 = (\sigma^2 + \nu\theta^2)T$ $c_4 = 3(\sigma^4\nu + 2\theta^4\nu^3 + 4\sigma^2\theta^2\nu^2)T$ $w = \frac{1}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2)$
Heston	$c_1 = \mu T + (1 - e^{\kappa T}) \frac{\theta - v_0}{2\kappa} - \frac{1}{2}\theta T$ $c_2 = \frac{1}{8\kappa^3}(\sigma T \kappa e^{-\kappa T}(v_0 - \theta)(8\kappa\rho - 4\sigma) + \kappa\rho\sigma(1 - e^{-\kappa T})(16\theta - 8v_0)$ $+ 2\theta\kappa T(-4\kappa\rho\sigma + \sigma^2 + 4\kappa^2)$ $+ \sigma^2((\theta - 2v_0)e^{-2\kappa T} + \theta(6e^{-\kappa T} - 7) + 2v_0) + 8\kappa^2(v_0 - \theta)(1 - e^{\kappa T}))$ $w = 0$

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